J. Differential Equations 248 (2010) 95–110

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Zero dissipation limit to strong contact discontinuity for the 1-D compressible Navier–Stokes equations

Shixiang Ma [a,b,*,1]

[a] *The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong*
[b] *School of Mathematical Sciences, South China Normal University, Guang Zhou 510631, China*

A R T I C L E   I N F O

Article history:
Received 23 March 2009
Revised 18 August 2009
Available online 5 September 2009

Keywords:
Compressible Navier–Stokes equations
Compressible Euler system
Zero dissipation limit
Contact discontinuity

A B S T R A C T

In this paper, we study the zero dissipation limit problem for the one-dimensional compressible Navier–Stokes equations. We prove that if the solution of the inviscid Euler equations is piecewise constants with a contact discontinuity, then there exist smooth solutions to the Navier–Stokes equations which converge to the inviscid solution away from the contact discontinuity at a rate of $\kappa^{\frac{1}{4}}$ as the heat-conductivity coefficient κ tends to zero, provided that the viscosity μ is of higher order than the heat-conductivity κ. Without loss of generality, we set $\mu \equiv 0$. Here we have no need to restrict the strength of the contact discontinuity to be small.

1. Introduction

The purpose of this paper is to study the asymptotic equivalence between the solutions of the compressible Navier–Stokes equations and those of the compressible Euler system when the viscosity and heat-conductivity coefficients are of different orders. The one-dimensional compressible Navier–Stokes equations in Lagrangian coordinates are expressed as

* Address for correspondence: School of Mathematical Sciences, South China Normal University, Guang Zhou 510631, China.
E-mail address: mashx822@gmail.com.

[1] This research is supported by Zheng Ge Ru Foundation, Hong Kong RGC Earmarked Research Grants CUHK4040/06P, CUHK4042/08P, RGC Central Allocation Grant, and NSFC 10501015.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.08.016

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu\left(\dfrac{u_x}{v}\right)_x, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = \kappa\left(\dfrac{\theta_x}{v}\right)_x + \mu\left(\dfrac{uu_x}{v}\right)_x, \quad x \in R,\ t > 0, \end{cases} \tag{1.1}$$

where v, u, θ, p and e denote the specific volume, the velocity, the temperature, the pressure, and the internal energy, respectively, and μ, κ are the viscosity and heat-conductivity coefficients, respectively. Here x is the Lagrangian coordinate, so that $x = constant$ corresponds to a particle path. We study the ideal polytropic gas, so that the pressure p and the internal energy e are related with v and θ by the following equations of state

$$p \equiv p(v, \theta) = R\theta/v, \qquad e \equiv e(\theta) = R\theta/(\gamma - 1) + constant, \tag{1.2}$$

where $R > 0$ is the gas constant and $\gamma \in (1, 2]$ is the adiabatic exponent.

For perfect fluids, that is, $\kappa = \mu = 0$, (1.1) becomes the well-known compressible Euler system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = 0, \quad x \in R,\ t > 0, \end{cases} \tag{1.3}$$

which is one of the most important nonlinear strictly hyperbolic systems of conservation laws.

Since the inviscid system (1.3) is an idealization when the dissipative effects are neglected, it is of great importance to study the asymptotic equivalence between the viscous flows and the corresponding inviscid flows in the limit of small dissipation. Indeed, there have been great interest and intensive studies. For the general viscous conservation laws with positive definite viscosity matrix, Bianchini and Bressan [1] considered the general solutions with the initial data having small total variations, they proved the convergence of the solutions for the viscous systems to those for the associated hyperbolic systems by establishing the uniform total variation estimates. Yet their method cannot be applied to the systems with general viscosity approximations, such as (1.1), whose viscosity matrix is only semi-positive definite and thus less dissipative. This remains an important open problem. However, there are also many significant works on special solutions. For the case that the Euler flow contains a single shock, Hoff and Liu [4] studied the isentropic case, they established the limit process from the solutions of the compressible Navier–Stokes equations to the single shock-wave solution of the corresponding compressible Euler system (so-called p-system). They show that the solutions to the isentropic Navier–Stokes equations with shock data exist and converge to the inviscid shocks as the viscosity vanishes, uniformly away from the shocks. Ignoring the initial layers, Goodman and Xin [2] gave a very detailed description of the asymptotic behavior of solutions for the general viscous systems as the viscosity tends to zero, via a method of matching asymptotics. This method can be applied to the Navier–Stokes equations (1.1). Later Yu [16] revealed the rich structure of nonlinear wave interactions due to the presence of shocks and initial layers by a detailed pointwise analysis. As far as rarefaction wave is concerned, Xin in [15] has obtained that the solutions for the isentropic Navier–Stokes equations with weak centered rarefaction wave data exist for all time and converge to the weak centered rarefaction wave solution of the corresponding Euler system, as the viscosity tends to zero, uniformly away from the initial discontinuity. Moreover, in the case that either the initial layers are ignored or the rarefaction waves are smooth, he also obtains a rate of convergence which is valid uniformly for all time. Recently Jiang et al. [8] improved the first part with weak centered rarefaction waves data and Zeng [17] improved the other results, respectively, in [15] to the full compressible Navier–Stokes equations, provided that the viscosity and heat-conductivity coefficients are in the same order. Furthermore, by a spectral analysis and Evans function method, Kevin Zumbrun and his collaborators have obtained many important results even for large amplitude and multi-dimensional case [3,11–13,18], etc. However, the case that the solutions to the Euler system containing contact discontinuity is much more subtle, which are the target of this paper. We expect to investigate the behaviors of the one-dimensional Navier–Stokes equations as the viscosity and heat-conductivity coefficients are small when the underlying inviscid flow contains contact discontinuities, with the help of recent significant works on nonlinear stability of contact discontinuity by Huang, Matsumura and Xin [7] and Huang, Li and Matsumura [6].

In this paper, we consider the case that the viscosity coefficient is of a higher order than the heat-conductivity coefficient. Without loss of generality, we assume that the viscosity μ is zero. Then the system (1.1) becomes

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = \kappa\left(\dfrac{\theta_x}{v}\right)_x, \quad x \in R,\ t > 0. \end{cases} \tag{1.4}$$

For the Riemann problem to the corresponding Euler system (1.3) with the Riemann initial data

$$(v, u, \theta)(x, 0) = \begin{cases} (v_-, u_-, \theta_-), & \text{if } x < 0, \\ (v_+, u_+, \theta_+), & \text{if } x > 0, \end{cases} \tag{1.5}$$

a contact discontinuity takes the form

$$(\tilde{V}, \tilde{U}, \tilde{\Theta})(x, t) = \begin{cases} (v_-, u_-, \theta_-), & \text{if } x < 0, \\ (v_+, u_+, \theta_+), & \text{if } x > 0, \end{cases} \tag{1.6}$$

provided that

$$u_- = u_+, \qquad p_- \equiv \frac{R\theta_-}{v_-} = \frac{R\theta_+}{v_+} \equiv p_+. \tag{1.7}$$

As in [7], in the setting of the compressible Navier–Stokes equations (1.4), the corresponding wave to the contact discontinuity becomes smooth and behaves as a diffusion wave due to the dissipation effect. We call this wave "viscous contact wave." We now construct the viscous contact wave $(\bar{v}, \bar{u}, \bar{\theta})$ as follows. Since the pressure of the profile $(\bar{v}, \bar{u}, \bar{\theta})$ is expected to be almost constant, that is,

$$\bar{p} \equiv \frac{R\bar{\theta}}{\bar{v}} \approx p_+, \tag{1.8}$$

which indicates that the energy equation $(1.4)_3$ is

$$\frac{R}{\gamma - 1}\theta_t + p_+ u_x = \kappa\left(\frac{\theta_x}{v}\right)_x. \tag{1.9}$$

Substituting (1.8) into (1.9) and using $(1.4)_1$ yield a nonlinear diffusion equation

$$\theta_t = a\kappa\left(\frac{\theta_x}{\theta}\right)_x, \qquad \theta(-\infty, t) = \theta_-, \qquad \theta(+\infty, t) = \theta_+, \qquad a = \frac{p_+(\gamma - 1)}{\gamma R^2} > 0, \tag{1.10}$$

which admits a unique self-similar solution $\Theta(x, t) = \Theta(\xi)$, $\xi = \frac{x}{\sqrt{1+t}}$ due to [5,14]. Furthermore, $\Theta(\xi)$ is a monotone function, increasing if $\theta_+ > \theta_-$ and decreasing if $\theta_+ < \theta_-$. Let $\delta = |\theta_+ - \theta_-|$, then Θ satisfies

$$\left|\left(\kappa(1+t)\right)^{\frac{l}{2}} \partial_x^l \Theta\right| + |\Theta - \theta_\pm| \leqslant c_1 \delta e^{-\frac{c_2 x^2}{\kappa(1+t)}} \quad \text{as } |x| \to \infty,\ l \geqslant 1. \tag{1.11}$$

With Θ so determined, we can define the contact wave profile $(\bar{v}, \bar{u}, \bar{\theta})$ as follows:

$$\bar{v} = \frac{R}{p_+}\Theta, \qquad \bar{u} = u_- + \frac{(\gamma-1)\kappa}{\gamma R}\frac{\Theta_x}{\Theta}, \qquad \bar{\theta} = \Theta. \tag{1.12}$$

Then $(\bar{v}, \bar{u}, \bar{\theta})$ satisfies

$$\|\bar{v} - \tilde{V}, \bar{u} - \tilde{U}, \bar{\theta} - \tilde{\Theta}\|_{L^p} = O\big(\kappa^{1/(2p)}\big)(1+t)^{1/(2p)}, \quad p \geqslant 1, \tag{1.13}$$

and

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ \bar{u}_t + \bar{p}_x = \bar{R}_1, \\ \left(\bar{e} + \dfrac{\bar{u}^2}{2}\right)_t + (\bar{u}\bar{p})_x = \kappa\left(\dfrac{\bar{\theta}_x}{\bar{v}}\right)_x + \bar{u}\bar{R}_1, \end{cases} \tag{1.14}$$

where $\bar{e} = \frac{R\bar{\theta}}{\gamma-1}$ and

$$\bar{R}_1 = \frac{(\gamma-1)\kappa}{\gamma R}(\ln\Theta)_{xt} = O(\delta)\kappa^2\big(\kappa(1+t)\big)^{-\frac{3}{2}}e^{-\frac{c_2x^2}{\kappa(1+t)}} \quad \text{as } |x| \to \infty. \tag{1.15}$$

The main results of this paper are as follows:

Theorem 1.1. *For any given (v_-, u_-, θ_-), suppose that (v_+, u_+, θ_+) satisfies (1.7). Let $(\tilde{V}, \tilde{U}, \tilde{\Theta})$ be a contact discontinuity solution of the form (1.6) with finite strength to the Euler system (1.3). Then, there exists constant $\kappa_0 > 0$, such that for each $\kappa \in (0, \kappa_0]$, there is a smooth solution $(v^\kappa, u^\kappa, \theta^\kappa)$ to (1.4) on $R \times R^+$, still denoted by (v, u, θ), with the same initial data as $(\bar{v}, \bar{u}, \bar{\theta})$. Moreover, for any arbitrarily large $T > 0$ and small $h > 0$, it holds that*

$$\sup_{0\leqslant t\leqslant T}\int \big|(v,u,\theta)(x,t) - (\tilde{V}, \tilde{U}, \tilde{\Theta})(x,t)\big|^2\,dx \leqslant C\kappa^{\frac{1}{2}}, \tag{1.16}$$

and

$$\sup_{0\leqslant t\leqslant T,\,|x|\geqslant h}\big|(v,u,\theta)(x,t) - (\tilde{V}, \tilde{U}, \tilde{\Theta})(x,t)\big| \leqslant C\kappa^{\frac{1}{4}}, \tag{1.17}$$

where C is a positive constant independent of κ.

Remark 1.2. This is the first paper to deal with the zero dissipation limit result for contact discontinuities. Furthermore, here we only require the strength of the contact discontinuity to be finite and have no need to restrict it to be small.

Remark 1.3. The convergence rate in (1.17) may not be optimal. We conjecture that it can be improved to be $\kappa^{\frac{1}{2}}$. This will be left for future study.

Remark 1.4. The essential difficulty of this paper is that we deal with the case that the viscosity μ can be higher order than the heat-conductivity κ. Here we suppose that $\mu \equiv 0$, so the system is much less dissipation. Formally, the first two equations of the Navier–Stokes system (1.4) are hyperbolic and only the last one is parabolic, so how to derive the suitable a priori estimates will be difficult. Explicitly, how to control the term $\int \|(\phi_y, \psi_y)(\cdot,\tau)\|\,d\tau$ (see (2.1))? If we directly do the energy estimates, we will find that we can only use $\int \|\psi_y(\cdot,\tau)\|\,d\tau$ to control $\int \|\phi_y(\cdot,\tau)\|\,d\tau$. Yet, here $\mu \equiv 0$, we cannot directly get the a priori estimates for $\int \|\psi_y(\cdot,\tau)\|\,d\tau$. Thus, we must think of a method so that we can use $\int \|\zeta_y(\cdot,\tau)\|\,d\tau$ to control the both terms $\int \|\phi_y(\cdot,\tau)\|\,d\tau$ and $\int \|\psi_y(\cdot,\tau)\|\,d\tau$. That is the work in Section 4 of our paper, motivated by [9] for constant perturbation, and cannot be derived from the cited papers by Huang and coauthors. When $\mu \neq 0$ and $\mu = o(\kappa)$, by scaling with κ, we can find that the coefficient of the term $\int \|\psi_y(\cdot,\tau)\|\,d\tau$ will be $\frac{\mu}{\kappa}$, which tends to zero. This is similar as the case $\mu \equiv 0$. If the viscosity μ is of the same order as the heat-conductivity κ, by direct energy estimates, we can control the term $\int \|\psi_y(\cdot,\tau)\|\,d\tau$ and then control $\int \|\phi_y(\cdot,\tau)\|\,d\tau$. Thus by a more simple proof, we can obtain the similar results.

Notation: In this paper, $|a| = (\sum_{i=1}^n a_i^2)^{\frac{1}{2}}$ if $a = (a_i)$ is a vector in R^n and $|A| = (\sum_{i=1}^n \sum_{j=1}^n A_{ij}^2)^{\frac{1}{2}}$ if $A = (A_{ij})_{n\times n}$ is a matrix. We also use H^l $(l \geqslant 1)$ to denote the usual Sobolev space with the norm $\|\cdot\|_l$ and $\|\cdot\| = \|\cdot\|_0$ denotes the usual L^2-norm.

2. Reformulation of the problem

Due to the estimates (1.11) and (1.13), to prove the main theorem, it suffices to show that there exists an exact solution to (1.4) in a neighborhood of the approximate solution $\bar{U} \equiv (\bar{v},\bar{u},\bar{\theta})$, and that the asymptotic behavior of the solution to (1.4) is given by $\bar{U}$ for small heat-conductivity κ.

Suppose that $U \equiv (v,u,\theta)$ is the exact solution to (1.4) with the initial data $U(x,0) = \bar{U}(x,0)$. We decompose the solution as

$$\phi = v - \bar{v}, \qquad \psi = u - \bar{u}, \qquad \zeta = \theta - \bar{\theta}. \tag{2.1}$$

Then using the relation (1.14) for $\bar{U}$, we obtain that

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + \left(\dfrac{R\zeta - p_+\phi}{v}\right)_x = -\bar{R}_1, \\ \dfrac{R}{\gamma-1}\zeta_t + pu_x - p_+\bar{u}_x = \kappa\left(\dfrac{\theta_x}{v} - \dfrac{\bar{\theta}_x}{\bar{v}}\right)_x, \\ \phi(x,0) = \psi(x,0) = \zeta(x,0) = 0. \end{cases} \tag{2.2}$$

Using the following scalings,

$$y = \frac{x}{\kappa}, \qquad \tau = \frac{1+t}{\kappa}, \tag{2.3}$$

we transform (2.2) into

$$\begin{cases} \phi_\tau - \psi_y = 0, \\ \psi_\tau + \left(\dfrac{R\zeta - p_+\phi}{v}\right)_y = -R_1, \\ \dfrac{R}{\gamma-1}\zeta_\tau + pu_y - p_+\bar{u}_y = \left(\dfrac{\theta_y}{v} - \dfrac{\bar{\theta}_y}{\bar{v}}\right)_y, \\ \phi(y,\tau_0) = \psi(y,\tau_0) = \zeta(y,\tau_0) = 0, \end{cases} \tag{2.4}$$

where $\tau_0 = 1/\kappa$, $R_1 = \kappa\bar{R}_1$ and

$$\left|\partial_y^l \Theta\right| \leqslant c_1\kappa^{\frac{l}{2}} e^{-\frac{c_2 y^2}{\tau}}, \quad l \geqslant 1; \qquad |R_1| \leqslant c_1\kappa^{\frac{3}{2}} e^{-\frac{c_2 y^2}{\tau}}. \tag{2.5}$$

Set $\tau_1 = \frac{1+T}{\kappa}$. Then we only need to show that for suitably small κ, (2.4) has a unique "small" smooth solution on $R \times [\tau_0, \tau_1]$. By the standard existence and uniqueness theory, and the continuous induction argument for hyperbolic–parabolic equations [9], it suffices to close the following a priori estimate

$$N(\tau) \equiv \left\|(\phi, \psi, \zeta)(\cdot, \tau)\right\|_2 \leqslant \varepsilon, \tag{2.6}$$

where ε is a positive small constant depending on T, the initial data and the strength of the contact discontinuity. This is a consequence of a series of lemmas. We start with the lower order estimate.

3. Lower order estimate

Lemma 3.1. *Suppose that the Cauchy problem* (2.4) *has a solution* $(\phi, \psi, \zeta) \in C^1([\tau_0, \tau_2] : H^2(R^1))$ *for some* $\tau_0 < \tau_2 < \tau_1$. *Then there exist positive constants* ε_1, κ_1 *and* c, *which are independent of* κ *and* τ_2, *such that if* $0 < \varepsilon < \varepsilon_1$ *and* $\kappa \leqslant \kappa_1$, *we have*

$$\sup_{\tau_0 \leqslant \tau \leqslant \tau_1} \left\|(\phi, \psi, \zeta)(\cdot, \tau)\right\|^2 + \int_{\tau_0}^{\tau_1} \left\|\zeta_y(\cdot, \tau)\right\|^2 d\tau \leqslant c\kappa^{\frac{1}{2}}. \tag{3.1}$$

Proof. Similar to [6], we have

$$\left(\frac{1}{2}\psi^2 + R\bar{v}\Phi\left(\frac{v}{\bar{v}}\right) + \frac{R}{\gamma - 1}\bar{\theta}\Phi\left(\frac{\theta}{\bar{\theta}}\right)\right)_\tau + \frac{1}{v\theta}\zeta_y^2 + L_y + Q = R_1\psi, \tag{3.2}$$

where

$$\Phi(s) = s - 1 - \ln s, \tag{3.3}$$

$$L = R\left(\frac{\theta}{v} - \frac{\bar{\theta}}{\bar{v}}\right)\psi - \frac{\zeta}{\theta}\left(\frac{\theta_y}{v} - \frac{\bar{\theta}_y}{\bar{v}}\right), \tag{3.4}$$

and

$$Q = p_+\Phi\left(\frac{\bar{v}}{v}\right)\bar{u}_y + \frac{p_+}{\gamma - 1}\Phi\left(\frac{\bar{\theta}}{\theta}\right)\bar{u}_y - \frac{\zeta}{\theta}(p_+ - p)\bar{u}_y - \frac{\theta_y}{\theta^2 v}\zeta\zeta_y - \frac{\zeta_y\phi}{\theta v\bar{v}}\bar{\theta}_y + \frac{\theta_y\zeta\phi}{\theta^2 v\bar{v}}\bar{\theta}_y, \tag{3.5}$$

satisfying

$$|Q| \leqslant (\varepsilon + \eta)\zeta_y^2 + c_\eta\left(|\Theta_{yy}| + |\Theta_y|^2\right)\left(\phi^2 + \zeta^2\right), \tag{3.6}$$

where $\eta > 0$ is a constant to be determined later. Then (3.2)–(3.6) and (2.5) yield that

$$\begin{aligned}
&\int \left(\frac{1}{2}\psi^2 + R\bar{v}\Phi\left(\frac{v}{\bar{v}}\right) + \frac{R}{\gamma - 1}\bar{\theta}\Phi\left(\frac{\theta}{\bar{\theta}}\right)\right) dy + \int_{\tau_0}^{\tau}\int \frac{1}{v\theta}\zeta_y^2\, dy\, d\tau \\
&\quad = \int_{\tau_0}^{\tau}\int (-Q + R_1\psi)\, dy\, d\tau \\
&\quad \leqslant c(\varepsilon + \eta)\int_{\tau_0}^{\tau}\int \zeta_y^2\, dy\, d\tau + c_\eta\kappa \int_{\tau_0}^{\tau}\int \left(\phi^2 + \psi^2 + \zeta^2\right) dy\, d\tau + \kappa^{-1}\int_{\tau_0}^{\tau}\int R_1^2\, dy\, d\tau.
\end{aligned} \tag{3.7}$$

Using (2.5) and taking ε and η to be sufficiently small, we obtain

$$\left\|\left(\phi^2+\psi^2+\zeta^2\right)(\cdot,\tau)\right\|^2+\int_{\tau_0}^{\tau}\left\|\zeta_y(\cdot,\tau)\right\|^2 d\tau \leqslant c\kappa\int_{\tau_0}^{\tau}\left\|\left(\phi^2+\psi^2+\zeta^2\right)(\cdot,\tau)\right\|^2+c\kappa^{\frac{1}{2}}. \quad (3.8)$$

And then we apply Gronwall's inequality to deduce that

$$\left\|\left(\phi^2+\psi^2+\zeta^2\right)(\cdot,\tau)\right\|^2+\int_{\tau_0}^{\tau}\left\|\zeta_y(\cdot,\tau)\right\|^2 d\tau \leqslant c\kappa^{\frac{1}{2}}. \quad (3.9)$$

This finishes the proof of Lemma 3.1. $\square$

4. Higher order estimates

Lemma 4.1. *Suppose that the conditions in Lemma* 3.1 *are satisfied. Then*

$$\left\|(\phi_y,\psi_y,\zeta_y)(\cdot,\tau)\right\|^2+\int_{\tau_0}^{\tau}\left(\left\|(\phi_y,\psi_y)(\cdot,\tau)\right\|^2+\left\|\zeta_{yy}(\cdot,\tau)\right\|^2\right)d\tau \leqslant c\kappa^{\frac{1}{2}}, \quad (4.1)$$

for all $\tau\in[\tau_0,\tau_2]$, where the constant c is independent of τ_2 and κ.

Proof. *Step 1*. Rewrite (1.4) in the following symmetric form

$$A^0(U)U_\tau+A(U)U_y=B(U)U_{yy}+g(U,U_y), \quad (4.2)$$

where $g(U,U_y)=(0,0,\theta_y(\frac{1}{v})_y)^t$, and

$$A^0(U)=\begin{pmatrix}-\theta p_v & 0 & 0\\ 0 & \theta & 0\\ 0 & 0 & \frac{R}{\gamma-1}\end{pmatrix},\qquad A(U)=\begin{pmatrix}0 & \theta p_v & 0\\ \theta p_v & 0 & p\\ 0 & p & 0\end{pmatrix},\qquad B(U)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&\frac{1}{v}\end{pmatrix}.$$

Consequently, the system (1.14) is transformed into

$$A^0(\bar U)\bar U_\tau+A(\bar U)\bar U_y=B(\bar U)\bar U_{yy}+g(\bar U,\bar U_y)+\bar F, \quad (4.3)$$

where $\bar F=(0,\bar\theta R_1,0)^t$. Now we define a new matrix $\tilde A(U)$ as

$$\tilde A(U)=\begin{pmatrix}A_{11}(U) & A_{12}(\bar U)\\ A_{21}(\bar U) & 0\end{pmatrix}, \quad (4.4)$$

where $A_{11}(U)=\left(\begin{smallmatrix}0 & \theta p_v\\ \theta p_v & 0\end{smallmatrix}\right)$ and $A_{12}(U)=\left(\begin{smallmatrix}0\\ p\end{smallmatrix}\right)=A_{21}(U)^t$. Set $W=U-\bar U$. (4.2)–(4.4) lead to

$$A^0(U)W_\tau+\tilde A(U)W_y=B(U)W_{yy}+\tilde g(U,U_y)+\big(\tilde A(U)-A(U)\big)W_y, \quad (4.5)$$

where

$$\tilde{g}(U,U_y)=\big\{\big(A^0(\bar{U})-A^0(U)\big)\bar{U}_\tau+\big(B(U)-B(\bar{U})\big)\bar{U}_{yy}\big\}+\big(A(\bar{U})-A(U)\big)\bar{U}_y$$
$$+\big(g(U,U_y)-g(\bar{U},\bar{U}_y)\big)-\bar{F}.$$

Differentiating (4.5) with respect to y, multiplying the resulting system by $\partial_y W$ and integrating on R, we obtain

$$\int\big\langle A^0(U)\partial_y W_\tau,\partial_y W\big\rangle dy+\int\big\langle\tilde{A}(U)\partial_y^2 W,\partial_y W\big\rangle dy$$
$$=\int\big\langle B(U)\partial_y^3 W,\partial_y W\big\rangle dy+\int\langle\tilde{H},\partial_y W\rangle\,dy. \tag{4.6}$$

Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product on R^3, and

$$\tilde{H}=A^0(U)\partial_y\big(A^0(U)^{-1}\tilde{g}\big)+A^0(U)\big[\partial_y,A^0(U)^{-1}B(U)\big]W_{yy}$$
$$+A^0(U)\partial_y\big\{A^0(U)^{-1}\big(\tilde{A}(U)-A(U)\big)W_y\big\}-A^0(U)\big[\partial_y,A^0(U)^{-1}\tilde{A}(U)\big]W_y,$$

where $[\cdot,\cdot]\cdot$ denotes the commutator. Next we will estimate the terms in (4.6) separately. First, using (1.12), (2.1), (2.5) and the system (2.4), we have

$$\int\big\langle A^0(U)\partial_y W_\tau,\partial_y W\big\rangle dy=\frac{1}{2}\frac{d}{d\tau}\int\big\langle A^0(U)\partial_y W,\partial_y W\big\rangle dy-\frac{1}{2}\int\big\langle\partial_\tau A^0(U)\partial_y W,\partial_y W\big\rangle dy$$
$$\geqslant\frac{1}{2}\frac{d}{d\tau}\int\big\langle A^0(U)\partial_y W,\partial_y W\big\rangle dy-c(\varepsilon+\kappa)\int\big(\phi_y^2+\psi_y^2+\zeta_{yy}^2\big)\,dy.$$

Similarly, Sobolev's inequality and Young's inequality yield

$$-\int\big\langle\tilde{A}(U)\partial_y^2 W,\partial_y W\big\rangle dy=\frac{1}{2}\int\big\langle\partial_y\tilde{A}(U)\partial_y W,\partial_y W\big\rangle dy$$
$$\leqslant c\int\big(|W_y|+|\bar{U}_y|\big)|W_y|^2\,dy$$
$$\leqslant c\big(\varepsilon+\kappa^{\frac{1}{2}}\big)\int\big(\phi_y^2+\psi_y^2+\zeta_y^2\big)\,dy.$$

By a direct calculation, the third term is estimated as

$$\int\big\langle B(U)\partial_y^3 W,\partial_y W\big\rangle dy=\int\frac{1}{v}\partial_y^3\zeta\,\partial_y\zeta\,dy$$
$$=-\int\frac{1}{v}\big(\partial_y^2\zeta\big)^2\,dy+\int\frac{1}{v^2}(\phi_y+\bar{v}_y)\partial_y^2\zeta\,\partial_y\zeta\,dy$$
$$\leqslant-\int\frac{1}{v}\big(\partial_y^2\zeta\big)^2\,dy+c\big(\varepsilon+\kappa^{\frac{1}{2}}\big)\int\big(\big(\partial_y^2\zeta\big)^2+\zeta_y^2\big)\,dy.$$

Finally,

$$\begin{aligned}\int \langle \tilde{H}, \partial_y W\rangle\, dy = {} & \int \bigl\langle A^0(U)\partial_y\bigl(A^0(U)^{-1}\tilde{g}\bigr), \partial_y W\bigr\rangle dy \\ & + \int \bigl\langle A^0(U)\bigl[\partial_y, A^0(U)^{-1}B(U)\bigr]W_{yy}, \partial_y W\bigr\rangle dy \\ & + \int \bigl\langle A^0(U)\partial_y\bigl\{A^0(U)^{-1}\bigl(\tilde{A}(U) - A(U)\bigr)W_y\bigr\} \\ & - A^0(U)\bigl[\partial_y, A^0(U)^{-1}\tilde{A}(U)\bigr]W_y, \partial_y W\bigr\rangle dy.\end{aligned}$$

We denote the terms on the right in order by I, II, III, which can be estimated separately below.

$$I = \int \bigl\langle A^0(U)\partial_y\bigl(A^0(U)^{-1}\bigr)\tilde{g}, \partial_y W\bigr\rangle dy + \int \langle \partial_y \tilde{g}, \partial_y W\rangle\, dy \equiv I_1 + I_2.$$

Since

$$\partial_y\bigl(A^0(U)^{-1}\bigr) = \begin{pmatrix} -\partial_y((\theta p_v)^{-1}) & 0 & 0 \\ 0 & \partial_y(\theta^{-1}) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $\partial_y(A^0(U)^{-1})(g(U, U_y) - g(\bar{U}, \bar{U}_y))$ is zero vector in R^3. And then using the estimates in (2.5) and Lemma 3.1, we have

$$\begin{aligned} I_1 & \leqslant c\int \bigl(|W_y| + |\bar{U}_y|\bigr)\bigl\{\bigl(|\bar{U}_\tau| + |\bar{U}_{yy}| + |\bar{U}_y|\bigr)|W| + |\bar{F}|\bigr\}|W_y|\, dy \\ & \leqslant c\kappa^{\frac{1}{2}}\int |W_y|^2\, dy + c\kappa^{\frac{3}{2}}\int \bigl|W(\cdot,\tau)\bigr|^2\, dy + c\kappa^{\frac{1}{2}}\int |R_1|^2\, dy \\ & \leqslant c\kappa^{\frac{1}{2}}\int \bigl(\phi_y^2 + \psi_y^2 + \zeta_y^2\bigr)\, dy + c\kappa^2,\end{aligned}$$

provided that $\|W(\cdot,\tau)\|_{L^\infty}$ is bounded.

$$\begin{aligned} I_2 = {} & \int \bigl\langle \partial_y\bigl\{\bigl(A^0(\bar{U}) - A^0(U)\bigr)\bar{U}_\tau + \bigl(B(U) - B(\bar{U})\bigr)\bar{U}_{yy} + \bigl(A(\bar{U}) - A(U)\bigr)\bar{U}_y\bigr\}, \partial_y W\bigr\rangle dy \\ & + \int \bigl\langle \partial_y\bigl(g(U, U_y) - g(\bar{U}, \bar{U}_y)\bigr), \partial_y W\bigr\rangle dy - \int \langle \partial_y \bar{F}, \partial_y W\rangle\, dy \\ \equiv {} & \sum_{j=1}^{3} I_{2j}.\end{aligned}$$

By the definition of $\bar{U}$ and the estimates (2.5) and Lemma 3.1 again, we get

$$\begin{aligned} I_{21} & \leqslant c\int \bigl(|\bar{U}_\tau| + |\bar{U}_{yy}| + |\bar{U}_y|\bigr)\bigl(|W_y| + |\bar{U}_y||W|\bigr)|W_y|\, dy \\ & \quad + c\int \bigl(|\bar{U}_{\tau y}| + \bigl|\partial_y^3\bar{U}\bigr| + \bigl|\partial_y^2\bar{U}\bigr|\bigr)|W||W_y|\, dy \\ & \leqslant c\kappa^{\frac{1}{2}}\int |W_y|^2\, dy + c\kappa^{\frac{3}{2}}\int \bigl|W^2\bigr|\, dy \\ & \leqslant c\kappa^{\frac{1}{2}}\int \bigl(\phi_y^2 + \psi_y^2 + \zeta_y^2\bigr)\, dy + c\kappa^2.\end{aligned}$$

I_{22} is estimated as follows:

$$
\begin{aligned}
I_{22} &= \int \partial_y \left\{ \left(\frac{1}{v}\right)_y \theta_y - \left(\frac{1}{\bar{v}}\right)_y \bar{\theta}_y \right\} \zeta_y \, dy \\
&= -\int \left\{ \left(\frac{1}{v}\right)_y \theta_y - \left(\frac{1}{\bar{v}}\right)_y \bar{\theta}_y \right\} \zeta_{yy} \, dy \\
&\leqslant c \int \bigl\{ |\phi_y||\zeta_y| + \bigl(|\bar{\theta}_y||\phi_y| + |\bar{v}_y||\zeta_y|\bigr) + |\bar{v}_y||\bar{\theta}_y||\phi| \bigr\} |\zeta_{yy}| \, dy \\
&\leqslant c\bigl(\varepsilon + \kappa^{\frac{1}{2}}\bigr) \int \bigl(\phi_y^2 + \zeta_y^2 + \zeta_{yy}^2\bigr) \, dy + c\kappa^{\frac{3}{2}} \int \phi^2 \, dy \\
&\leqslant c\bigl(\varepsilon + \kappa^{\frac{1}{2}}\bigr) \int \bigl(\phi_y^2 + \zeta_y^2 + \zeta_{yy}^2\bigr) \, dy + c\kappa^2.
\end{aligned}
$$

Finally, Young's inequality and the estimates in (2.5) yield that

$$
I_{23} \leqslant c\kappa \int |\partial_y W|^2 \, dy + c\kappa^{-1} \int \bigl|\partial_y(\bar{\theta} R_1)\bigr|^2 \, dy \leqslant c\kappa \int \bigl(\phi_y^2 + \psi_y^2 + \zeta_y^2\bigr) \, dy + c\kappa^{\frac{5}{2}}.
$$

Consequently,

$$
I_2 = \sum_{j=1}^{3} I_{2j} \leqslant c\bigl(\varepsilon + \kappa^{\frac{1}{2}}\bigr) \int \bigl(\phi_y^2 + \psi_y^2 + \zeta_y^2 + \zeta_{yy}^2\bigr) \, dy + c\kappa^2.
$$

And then

$$
I = I_1 + I_2 \leqslant c\bigl(\varepsilon + \kappa^{\frac{1}{2}}\bigr) \int \bigl(\phi_y^2 + \psi_y^2 + \zeta_y^2 + \zeta_{yy}^2\bigr) \, dy + c\kappa^{\frac{5}{2}}.
$$

We continue to estimate the terms II and III.

$$
\begin{aligned}
II &= \int \bigl\langle A^0(U) \partial_y \bigl(A^0(U)^{-1} B(U)\bigr) W_{yy}, W_y \bigr\rangle \, dy = \int \partial_y \left(\frac{1}{v}\right) \zeta_y \zeta_{yy} \, dy \\
&\leqslant c\bigl(\varepsilon + \kappa^{\frac{1}{2}}\bigr) \int \bigl(\zeta_y^2 + \zeta_{yy}^2\bigr) \, dy.
\end{aligned}
$$

Differentiating directly shows that

$$
\begin{aligned}
III &= \int \bigl\langle A^0(U) \partial_y \bigl(A^0(U)^{-1}\bigr) \bigl(\tilde{A}(U) - A(U)\bigr) W_y, W_y \bigr\rangle \, dy \\
&\quad + \int \bigl\langle \partial_y \bigl\{ \bigl(\tilde{A}(U) - A(U)\bigr) W_y \bigr\} - A^0(U) \partial_y \bigl\{ \bigl(A^0(U)^{-1}\bigr) \tilde{A}(U) \bigr\} W_y, W_y \bigr\rangle \, dy \\
&\equiv III_1 + III_2.
\end{aligned}
$$

First, Sobolev's inequality gives

$$
III_1 \leqslant c \int \bigl(|W_y| + |\bar{U}_y|\bigr) |W| |W_y|^2 \, dy \leqslant c\varepsilon \int |W_y|^2,
$$

provided that $\|W_y(\cdot,\tau)\|_{L^\infty}$ is bounded. Using integration by parts, we have

$$\begin{aligned}
III_2 &= \int \big\{\psi_y \partial_y\big((\bar p - p)\zeta_y\big) + \zeta_y \partial_y\big((\bar p - p)\psi_y\big)\big\}\, dy \\
&\quad - \int \left\{\theta\psi_y\left(\phi_y \partial_y p_v + \zeta_y \partial_y\left(\frac{\bar p}{\theta}\right)\right) + \partial_y \bar p \psi_y \zeta_y\right\} dy \\
&= \int \psi_y \partial_y(\bar p - p)\zeta_y\, dy - \int \left\{\theta\psi_y\left(\phi_y \partial_y p_v + \zeta_y \partial_y\left(\frac{\bar p}{\theta}\right)\right) + \partial_y \bar p \psi_y \zeta_y\right\} dy \\
&\leqslant c\big(\varepsilon + \kappa^{\frac12}\big) \int \big(\phi_y^2 + \psi_y^2 + \zeta_y^2\big)\, dy.
\end{aligned}$$

Hence it follows that

$$III \leqslant c\big(\varepsilon + \kappa^{\frac12}\big) \int \big(\phi_y^2 + \psi_y^2 + \zeta_y^2\big)\, dy.$$

And then

$$\int \langle \tilde H, \partial_y W\rangle\, dy = I + II + III \leqslant c\big(\varepsilon + \kappa^{\frac12}\big) \int \big(\phi_y^2 + \psi_y^2 + \zeta_y^2 + \zeta_{yy}^2\big)\, dy + c\kappa^2.$$

Collecting all the estimates we have obtained so far, after choosing ε and κ to be sufficiently small, we get

$$\frac{d}{d\tau}\int \big\langle A^0(U)\partial_y W, \partial_y W\big\rangle dy + \int \frac{1}{v}\zeta_{yy}^2\, dy \leqslant c\big(\varepsilon + \kappa^{\frac12}\big) \int \big(\phi_y^2 + \psi_y^2 + \zeta_y^2\big)\, dy + c\kappa^2.$$

Integrating this inequality with respect to τ and using Lemma 3.1, we arrive that

$$\big\|(\phi_y, \psi_y, \zeta_y)(\cdot,\tau)\big\|^2 + \int_{\tau_0}^{\tau} \big\|\zeta_{yy}(\cdot,\tau)\big\|^2 d\tau \leqslant c\big(\varepsilon + \kappa^{\frac12}\big) \int_{\tau_0}^{\tau} \big\|(\phi_y, \psi_y)(\cdot,\tau)\big\|^2 d\tau + c\kappa. \tag{4.7}$$

Step 2. In this step, we will estimate $\int_{\tau_0}^{\tau} \|(\phi_y, \psi_y)(\cdot,\tau)\|^2\, d\tau$. First, linearizing (4.2) at $\bar U$, and then subtracting (4.3) from the resulting system, one gets that

$$A^0(\bar U)W_\tau + A(\bar U)W_y = B(\bar U)W_{yy} + H, \tag{4.8}$$

where

$$\begin{aligned}
H = A^0(\bar U)\big\{&A^0(U)^{-1}g(U,U_y) - A^0(\bar U)^{-1}g(\bar U,\bar U_y) - \big(A^0(U)^{-1}A(U) - A^0(\bar U)^{-1}A(\bar U)\big)W_y \\
&- \big(A^0(U)^{-1}A(U) - A^0(\bar U)^{-1}A(\bar U)\big)\bar U_y + \big(A^0(U)^{-1}B(U) - A^0(\bar U)^{-1}B(\bar U)\big)W_{yy} \\
&+ \big(A^0(U)^{-1}B(U) - A^0(\bar U)^{-1}B(\bar U)\big)\bar U_{yy}\big\} - \bar F.
\end{aligned} \tag{4.9}$$

It follows from [10] that there is a real matrix S, such that SA^0 is skew-symmetric and $(SA)' + B$ is symmetric positive definite, where $(SA)'$ denotes the symmetric part of SA.

We multiply (4.8) by $W_y^t S(\bar U)$, and then integrate with respect to y on R to obtain

$$\begin{aligned}&\int \big\langle S(\bar U)A^0(\bar U)W_\tau, W_y\big\rangle dy + \int \big\langle S(\bar U)A(\bar U)W_y, W_y\big\rangle dy\\&\quad = \int \big\langle S(\bar U)B(\bar U)W_{yy}, W_y\big\rangle dy + \int \big\langle S(\bar U)H, W_y\big\rangle dy.\end{aligned}\tag{4.10}$$

Since SA^0 is skew-symmetric, the first term on the left of (4.10) can be written as

$$\begin{aligned}&\int \big\langle S(\bar U)A^0(\bar U)W_\tau, W_y\big\rangle dy\\&= \frac{1}{2}\bigg\{\frac{d}{d\tau}\int \big\langle S(\bar U)A^0(\bar U)W, W_y\big\rangle dy - \int \big\langle \big(S(\bar U)A^0(\bar U)\big)_\tau W, W_y\big\rangle dy\\&\quad + \int \big\langle \big(S(\bar U)A^0(\bar U)\big)_y W, W_\tau\big\rangle dy\bigg\}\\&= \frac{1}{2}\bigg\{\frac{d}{d\tau}\int \big\langle S(\bar U)A^0(\bar U)W, W_y\big\rangle dy - \int \big\langle \big(S(\bar U)A^0(\bar U)\big)_\tau W, W_y\big\rangle dy\\&\quad - \int \big\langle \big(S(\bar U)A^0(\bar U)\big)_y W, A^0(U)^{-1}A(U)W_y\big\rangle dy\\&\quad + \int \big\langle \big(S(\bar U)A^0(\bar U)\big)_y W, A^0(U)^{-1}B(U)W_{yy}\big\rangle dy\\&\quad + \int \big\langle \big(S(\bar U)A^0(\bar U)\big)_y W, A^0(U)^{-1}\tilde g(U, U_y)\big\rangle dy\bigg\},\end{aligned}$$

where the system (4.5) has been used. Substitute this into (4.10) to get

$$\begin{aligned}&\int \big\langle S(\bar U)A(\bar U)W_y, W_y\big\rangle dy\\&= -\frac{1}{2}\frac{d}{d\tau}\int \big\langle S(\bar U)A^0(\bar U)W, W_y\big\rangle dy + \frac{1}{2}\bigg\{\int \big\langle \big(S(\bar U)A^0(\bar U)\big)_\tau W, W_y\big\rangle dy\\&\quad + \int \big\langle \big(S(\bar U)A^0(\bar U)\big)_y W, A^0(U)^{-1}A(U)W_y\big\rangle dy\bigg\}\\&\quad - \frac{1}{2}\int \big\langle \big(S(\bar U)A^0(\bar U)\big)_y W, A^0(U)^{-1}B(U)W_{yy}\big\rangle dy\\&\quad - \frac{1}{2}\int \big\langle \big(S(\bar U)A^0(\bar U)\big)_y W, A^0(U)^{-1}\tilde g(U, U_y)\big\rangle dy\\&\quad + \int \big\langle S(\bar U)B(\bar U)W_{yy}, W_y\big\rangle dy + \int \big\langle S(\bar U)H, W_y\big\rangle dy.\end{aligned}\tag{4.11}$$

Next we estimate all of the terms above separately. First, the fact that $SA + B$ is symmetric positive definite implies that

$$\begin{aligned}\int \big\langle S(\bar U)A(\bar U)W_y, W_y\big\rangle dy &= \int \big\langle \big(S(\bar U)A(\bar U) + B(\bar U)\big)W_y, W_y\big\rangle dy - \int \big\langle B(\bar U)W_y, W_y\big\rangle dy\\&\geqslant a\int |W_y|^2\,dy - c\int \zeta_y^2\,dy,\end{aligned}$$

where $a > 0$ is a constant independent of κ. Using Young's inequality and Lemma 3.1, we have

$$\begin{aligned}
&\int \big\langle \big(S(\bar{U})A^0(\bar{U})\big)_\tau W, W_y\big\rangle dy + \int \big\langle \big(S(\bar{U})A^0(\bar{U})\big)_y W, A^0(U)^{-1}A(U)W_y\big\rangle dy \\
&\quad \leqslant c \int \big(|\bar{U}_\tau| + |\bar{U}_y|\big)|W||W_y|\,dy \\
&\quad \leqslant \eta \int |W_y|^2\,dy + c_\eta \kappa \int |W|^2\,dy \\
&\quad \leqslant \eta \int |W_y|^2\,dy + c_\eta \kappa^{\frac{3}{2}},
\end{aligned}$$

where $\eta > 0$ is a constant to be determined later. Due to the form of B, direct calculations and Young's inequality lead to

$$\begin{aligned}
&-\frac{1}{2}\int \big\langle \big(S(\bar{U})A^0(\bar{U})\big)_y W, A^0(U)^{-1}B(U)W_{yy}\big\rangle dy + \int \big\langle S(\bar{U})B(\bar{U})W_{yy}, W_y\big\rangle dy \\
&\quad \leqslant c \int |\bar{U}_y||W||\zeta_{yy}|\,dy + c\int |W_y||\zeta_{yy}|\,dy \\
&\quad \leqslant \eta \int |W_y|^2\,dy + c_\eta \int |\zeta_{yy}|^2\,dy + c\kappa \int |W|^2\,dy \\
&\quad \leqslant \eta \int |W_y|^2\,dy + c_\eta \int |\zeta_{yy}|^2\,dy + c\kappa^{\frac{3}{2}}.
\end{aligned}$$

It follows from the definition of $\tilde{g}$ and H that

$$\begin{aligned}
&-\int \big\langle \big(S(\bar{U})A^0(\bar{U})\big)_y W, A^0(U)^{-1}\tilde{g}(U, U_y)\big\rangle dy \\
&\quad \leqslant c\bigg\{ \int |\bar{U}_y|\big(|\bar{U}_\tau| + |\bar{U}_{yy}| + |\bar{U}_y|\big)|W|^2\,dy \\
&\qquad + \int |\bar{U}_y||W|\big(|W_y|^2 + |\bar{U}_y||W_y| + |\bar{U}_y|^2|W|\big)\,dy + \int |\bar{U}_y||W||R_1|\,dy\bigg\} \\
&\quad \leqslant c(\varepsilon+\kappa)\int |W_y|^2\,dy + c\kappa\int |W|^2\,dy + c\int |R_1|^2\,dy \\
&\quad \leqslant c(\varepsilon+\kappa)\int |W_y|^2\,dy + c\kappa^{\frac{3}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\int \big\langle S(\bar{U})H, W_y\big\rangle dy &\leqslant c\int \big\{\big(|W_y|^2 + |\bar{U}_y||W_y| + |\bar{U}_y|^2|W|\big) + |W||W_y| \\
&\qquad + |\bar{U}_y||W| + |W||\zeta_{yy}| + |\bar{U}_{yy}||W|\big\}|W_y|\,dy \\
&\leqslant c\big(\varepsilon + \kappa^{\frac{1}{2}} + \eta\big)\int |W_y|^2\,dy + c\varepsilon\int \zeta_{yy}^2\,dy + c_\eta\kappa\int |W|^2\,dy \\
&\leqslant c\big(\varepsilon + \kappa^{\frac{1}{2}} + \eta\big)\int |W_y|^2\,dy + c\varepsilon\int \zeta_{yy}^2\,dy + c_\eta\kappa^{\frac{3}{2}}.
\end{aligned}$$

Collecting all the estimates we have obtained, by choosing ε, κ and η to be sufficiently small, we get

$$a\int\left(\phi_y^2+\psi_y^2\right)dy\leqslant-\frac{d}{d\tau}\int\left\langle S(\bar U)A^0(\bar U)W,W_y\right\rangle dy+c\int\left(\zeta_y^2+\zeta_{yy}^2\right)dy+c\kappa^{\frac{3}{2}}.$$

Integrating this inequality with respect to τ and using Cauchy–Schwarz inequality and Lemma 3.1, we may conclude that

$$\int_{\tau_0}^{\tau}\!\!\int\left(\phi_y^2+\psi_y^2\right)dy\,d\tau\leqslant c\int\left(\phi_y^2+\psi_y^2+\zeta_y^2\right)dy+c\int_{\tau_0}^{\tau}\!\!\int\zeta_{yy}^2\,dy\,d\tau+c\kappa^{\frac{1}{2}}.\tag{4.12}$$

Multiplying a suitably small constant to (4.12), then adding the resulting inequality to (4.7) and taking ε and κ to be sufficiently small, we can obtain the estimate (4.1), which completes the proof of Lemma 4.1. □

For the second order derivatives, one has the following estimate.

Lemma 4.2. *Suppose that the conditions in Lemma* 3.1 *are satisfied. Then*

$$\left\|\left(\partial_y^2\phi,\partial_y^2\psi,\partial_y^2\zeta\right)(\cdot,\tau)\right\|^2+\int_{\tau_0}^{\tau}\left(\left\|\left(\partial_y^2\phi,\partial_y^2\psi\right)(\cdot,\tau)\right\|^2+\left\|\partial_y^3\zeta(\cdot,\tau)\right\|^2\right)d\tau\leqslant c\kappa^{\frac{1}{2}},\tag{4.13}$$

for all $\tau\in[\tau_0,\tau_2]$, where the constant c is independent of τ_2 and κ.

The proof is similar to the proof of Lemma 4.1. Hence we omit it.

5. Proof of Theorem 1.1

Combining the results of Lemma 3.1 and Lemmas 4.1–4.2 leads to

Proposition 5.1. *There exist positive constants κ_0 and C, which are independent of κ such that if $0<\kappa<\kappa_0$, then for any $T>0$, the Cauchy problem* (2.4) *has a unique solution $(\phi,\psi,\zeta)\in C^1([\frac{1}{\kappa},\frac{1+T}{\kappa}]:H^2(R^1))$. Furthermore, the following inequality holds*

$$\sup_{\frac{1}{\kappa}\leqslant\tau\leqslant\frac{1+T}{\kappa}}\left\|(\phi,\psi,\zeta)(\cdot,\tau)\right\|_2^2+\int_{\frac{1}{\kappa}}^{\frac{1+T}{\kappa}}\left(\left\|(\phi_y,\psi_y)(\cdot,\tau)\right\|_1^2+\left\|\zeta_y(\cdot,\tau)\right\|_2^2\right)d\tau\leqslant C\kappa^{\frac{1}{2}}.\tag{5.1}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For any $T>0$, in view of (5.1), we have

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}\int\left|(v-\bar v,u-\bar u,\theta-\bar\theta)(x,t)\right|^2dx\\&\quad=\kappa\sup_{\frac{1}{\kappa}\leqslant\tau\leqslant\frac{1+T}{\kappa}}\int\left|(v-\bar v,u-\bar u,\theta-\bar\theta)(y,\tau)\right|^2dy\leqslant C\kappa^{\frac{3}{2}}.\end{aligned}$$

Then it follows from this and (1.13) that

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\left\|(v-\tilde V,u-\tilde U,\theta-\tilde\Theta)(\cdot,t)\right\|^2\\
&\quad\leqslant\sup_{0\leqslant t\leqslant T}\left\|(v-\bar v,u-\bar u,\theta-\bar\theta)(\cdot,t)\right\|^2+\sup_{0\leqslant t\leqslant T}\left\|(\bar v-\tilde V,\bar u-\tilde U,\bar\theta-\tilde\Theta)(\cdot,t)\right\|^2\\
&\quad\leqslant C\kappa^{\frac12},
\end{aligned}$$

which gives (1.16). Finally,

$$\left\|(v-\bar v,u-\bar u,\theta-\bar\theta)(\cdot,t)\right\|_{L^\infty}\leqslant C\left\|(\phi,\psi,\zeta)(\cdot,t)\right\|^{\frac12}\left\|(\phi_y,\psi_y,\zeta_y)(\cdot,t)\right\|^{\frac12}\leqslant C\kappa^{\frac14}.$$

This, together with (1.11), yields (1.17).

Hence we have completed the proof of Theorem 1.1. □

Acknowledgments

This is part of the author's PhD thesis written under the supervision of professor Zhouping Xin at the Institute of Mathematical Sciences in the Chinese University of Hong Kong. The author would like to express the hearty thanks to Professor Xin for the motivation of this problem and the encouragement and guidance for this work. The author also thanks the referees for the good suggestions.

References

[1] Stefano Bianchini, Alberto Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, Ann. of Math. (2) 161 (1) (2005) 223–342.
[2] Jonathan Goodman, Zhou Ping Xin, Viscous limits for piecewise smooth solutions to systems of conservation laws, Arch. Ration. Mech. Anal. 121 (3) (1992) 235–265.
[3] Olivier Guès, Guy Métivier, Mark Williams, Kevin Zumbrun, Existence and stability of multidimensional shock fronts in the vanishing viscosity limit, Arch. Ration. Mech. Anal. 175 (2) (2005) 151–244.
[4] David Hoff, Tai-Ping Liu, The inviscid limit for the Navier–Stokes equations of compressible, isentropic flow with shock data, Indiana Univ. Math. J. 38 (4) (1989) 861–915.
[5] Ling Hsiao, Tai-Ping Liu, Nonlinear diffusive phenomena of nonlinear hyperbolic systems, Chinese Ann. Math. Ser. B 14 (4) (1993) 465–480; A Chinese summary appears in: Chinese Ann. Math. Ser. A 14 (6) (1993) 740.
[6] Feimin Huang, Jing Li, Akitaka Matsumura, Stability of combination of viscous contact wave with rarefaction waves for 1-D compressible Navier–Stokes system, Comm. Math. Phys., in press.
[7] Feimin Huang, Akitaka Matsumura, Zhouping Xin, Stability of contact discontinuities for the 1-D compressible Navier–Stokes equations, Arch. Ration. Mech. Anal. 179 (1) (2006) 55–77.
[8] Song Jiang, Guoxi Ni, Wenjun Sun, Vanishing viscosity limit to rarefaction waves for the Navier–Stokes equations of one-dimensional compressible heat-conducting fluids, SIAM J. Math. Anal. 38 (2) (2006) 368–384 (electronic).
[9] Shuichi Kawashima, Systems of hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics, Doctoral thesis, Kyoto Univ., 1983.
[10] Shuichi Kawashima, Large-time behaviour of solutions to hyperbolic–parabolic systems of conservation laws and applications, Proc. Roy. Soc. Edinburgh Sect. A 106 (1–2) (1987) 169–194.
[11] Corrado Mascia, Kevin Zumbrun, Stability of large-amplitude viscous shock profiles of hyperbolic–parabolic systems, Arch. Ration. Mech. Anal. 172 (1) (2004) 93–131.
[12] Corrado Mascia, Kevin Zumbrun, Stability of large-amplitude shock profiles of general relaxation systems, SIAM J. Math. Anal. 37 (3) (2005) 889–913 (electronic).
[13] Anders Szepessy, Kevin Zumbrun, Stability of rarefaction waves in viscous media, Arch. Ration. Mech. Anal. 133 (3) (1996) 249–298.
[14] C.J. van Duyn, L.A. Peletier, A class of similarity solutions of the nonlinear diffusion equation, Nonlinear Anal. 1 (3) (1976/77) 223–233.
[15] Zhou Ping Xin, Zero dissipation limit to rarefaction waves for the one-dimensional Navier–Stokes equations of compressible isentropic gases, Comm. Pure Appl. Math. 46 (5) (1993) 621–665.
[16] Shih-Hsien Yu, Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws, Arch. Ration. Mech. Anal. 146 (4) (1999) 275–370.

[17] Huihui Zeng, Asymptotic behavior of solutions to fluid dynamical equations, Doctoral thesis, The Chinese University of Hong Kong.
[18] Kevin Zumbrun, Peter Howard, Pointwise semigroup methods and stability of viscous shock waves, Indiana Univ. Math. J. 47 (3) (1998) 741–871.